

ANALYSIS OF EXPANDED MIXED FINITE ELEMENT METHODS FOR THE GENERALIZED FORCHHEIMER EQUATIONS

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Abstract. The nonlinear Forchheimer equations are used to describe the dynamics of fluid flows in porous media when Darcy's law is not applicable. In this article, we consider the generalized Forchheimer flows for slightly compressible fluids, and then study the expanded mixed finite element method applied to the initial boundary value problem for the resulting degenerate parabolic equation for pressure. The bounds for the solutions, time derivative and gradient of solutions are established. Utilizing the monotonicity properties of Forchheimer equation and boundedness of solutions, *a priori* error estimates for solution are obtained in L^2 -norm, L^∞ -norm as well as for its gradient in L^{2-a} -norm for all $a \in (0, 1)$. Optimal L^2 -error estimates are shown for solutions under some additional regularity assumptions. Numerical results using the lowest order Raviart-Thomas mixed element confirm the theoretical analysis regarding convergence rates.

Key words. Error estimates, expanded mixed finite element, nonlinear degenerate parabolic equations, generalized Forchheimer equations, porous media.

AMS subject classifications. 65M12, 65M15, 65M60, 35Q35, 76S05.

1. Introduction . Fluid flow in porous media is a great interest in many areas of reservoir engineering, such as petroleum, environmental and groundwater hydrology. Description of fluid flow behavior accurately in the porous media is essential to the successful design and operation of projects in these areas. Most of study of fluid flow in porous media are based on Darcy's law. By this law, the pressure gradient ∇p is linearly proportional to the fluid velocity \mathbf{u} in the porous media which writes as $\alpha \mathbf{u} = -\nabla p$ with empirical constant α . However Dupuit, a Darcy's student, observed on the field data that this linear relation is no longer valid for flows owning high velocity. A nonlinear relationship between velocity and gradient of pressure is introduced by adding the higher order term of velocity to the Darcy's law. It is known as Forchheimer laws. Engineers widely use the three following Forchheimer's laws (cf. [11]) to match experimental observation:

$$\alpha \mathbf{u} + \beta |\mathbf{u}| \mathbf{u} = -\nabla p, \quad \alpha \mathbf{u} + \beta |\mathbf{u}| \mathbf{u} + \lambda |\mathbf{u}|^2 \mathbf{u} = -\nabla p, \quad \alpha \mathbf{u} + \lambda_m |\mathbf{u}|^{m-1} \mathbf{u} = -\nabla p,$$

where $\alpha, \beta, \lambda, m, \lambda_m$ are empirical constants.

Since then, there is a large number of research on these equations and their variations, the Brinkman-Forchheimer equations for incompressible fluids (cf. [6, 7, 8, 12, 13, 26, 27, 28], see also [32]). Recently, study on slightly compressible fluid flows subject to generalized Forchheimer equations are in [3, 15, 16] and later in [17, 18, 19]. These are devoted to theory of existence, stability and qualitative property of solutions. The study of numerical methods for degenerate parabolic equations are still not analyzed as much as those of theory.

The popular numerical methods for modeling flow in porous media are the mixed finite element approximations in [9, 14, 21, 25] and block-centered finite difference method in [30] because these inherit conservation properties and produce the accurate flux (see [10]).

In [2] Arbogast, Wheeler and Zhang first analyzed mixed finite element approximations of degenerate parabolic equation arising in flow in porous media. Not so

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long later Arbogast, Wheeler and Yotov in [1] showed that the standard mixed finite element method not suitable for problems with small tensor coefficients as we need to invert the tensor. The proposed approach reduces original Forchheimer type equation to generalized Darcy equation with conductivity tensor K degenerating as gradient of the pressure convergence to infinity. At the same time, the standard mixed variational formulation requires inverting K to find gradient of pressure.

Woodward and Dawson in [33] study of expanded mixed finite element methods for a nonlinear parabolic equation modeling flow into variably saturated porous media. In their analysis, the Kirchhoff transformation is used to move the nonlinearity from coefficient K to the gradient and thus simplifies analysis of the equations. This transformation does not applicable for our system (2.8).

In this paper, we combine techniques developed in [15, 16] and the expanded mixed finite element method as in [1] to utilize both the special structures of equation as well as the advantages of the expanded mixed finite element method in obtaining the optimal order error estimates for the solution in several norms of interest.

The paper is organized as follows: In §2 we introduce the generalized formulation of the Forchheimers laws for slightly compressible fluids, recall the relevant results from [3, 15] and preliminary results. In §3 we consider the expanded mixed formulation and standard results for mixed finite element approximations. A implicit backward difference time discretization of the semidiscrete scheme is proposed to solve the system (3.3). In §4 we derive many bounds for solutions to (3.2) and (3.3) in Lebesgue norms. In §5 we analyze two version of a mixed finite element approximation, a semidiscrete version and a fully discrete version. The *priori* error estimates for the three relevant variables in L^2 -norms, L^∞ -norm are established. Under suitable assumptions on the regularity of solutions, we prove the superconvergence. In §6, we provide a numerical example using the lowest Raviart-Thomas mixed finite element. The results support our theoretical analysis regarding convergence rates.

2. Mathematical preliminaries and auxiliaries. We consider a fluid in a porous medium in a bounded domain $\Omega \subset \mathbb{R}^d, d \geq 2$. Its boundary $\Gamma = \partial\Omega$ belongs to C^2 . Let $x \in \mathbb{R}^d, 0 < T < \infty, t \in (0, T]$ be the spatial and time variable.

A general Forchheimer equation, which is studied in [3, 15, 17, 19] has the form

$$g(|\mathbf{u}|)\mathbf{u} = -\nabla p, \quad (2.1)$$

where $g(s) \geq 0$ is a function defined on $[0, \infty)$. When $g(s) = \alpha, \alpha + \beta s, \alpha + \beta s + \gamma s^2, \alpha + \gamma_m s^{m-1}$, where $\alpha, \beta, \gamma, m, \gamma_m$ are empirical constants, we have Darcy's law, Forchheimer's two term, three term and power laws, respectively. The function g in (2.1) is a polynomial with non-negative coefficients as the form

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + \dots + a_N s^{\alpha_N}, \quad s \geq 0, \quad (2.2)$$

where $N \geq 1, \alpha_0 = 0 < \alpha_1 < \dots < \alpha_N$ are fixed number, the coefficients a_0, \dots, a_N are non-negative numbers with $a_0 > 0, a_N > 0$. The number α_N is the degree of g is denoted by $\deg(g)$.

The monotonicity of the nonlinear term and the nondegeneracy of the Darcy's parts in (2.1) enable us to write \mathbf{u} implicit in terms of ∇p and derivative of a nonlinear Darcy equation:

$$\mathbf{u} = -K(|\nabla p|)\nabla p. \quad (2.3)$$

The function $K : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined by

$$K(\xi) = \frac{1}{g(s(\xi))} \text{ where } s = s(\xi) \geq 0 \text{ satisfies } sg(s) = \xi, \text{ for } \xi \geq 0. \quad (2.4)$$

The state equation, which relates the density $\rho(x, t) > 0$ with pressure p , for slightly compressible fluids is

$$\frac{d\rho}{dp} = \kappa^{-1} \rho \text{ or } \rho(p) = \rho_0 \exp\left(\frac{p - p_0}{\kappa}\right), \quad \kappa > 0. \quad (2.5)$$

Other equations governing the fluid's motion are the equation of continuity:

$$\frac{d\rho}{dt} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

which yields

$$\frac{d\rho}{dp} \frac{dp}{dt} + \rho \nabla \cdot \mathbf{u} + \frac{d\rho}{dp} \mathbf{u} \cdot \nabla p = 0. \quad (2.6)$$

Combining (2.6) and (2.5), we find that

$$\frac{dp}{dt} + \kappa \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p = 0. \quad (2.7)$$

Since for most slightly compressible fluids in porous media the value of the constant κ is large, following engineering tradition we drop the last term in (2.7) and study the reduced equation,

$$\frac{dp}{dt} + \kappa \nabla \cdot \mathbf{u} = 0. \quad (2.8)$$

By rescaling the time variable, hereafter we assume that $\kappa = 1$.

Let $\mathbf{s} = \nabla p$. Equations (2.8) and (2.3) are equivalent to the system

$$\begin{cases} p_t + \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u} + K(|\mathbf{s}|)\mathbf{s} = 0, \\ \mathbf{s} - \nabla p = 0. \end{cases} \quad (2.9)$$

The following properties of function $K(\xi)$ are proved in Lemma III.5 and III.9 of [3], Lemma 2.1 and 5.2 of [15].

LEMMA 2.1. *We have for any $\xi \geq 0$ that*

(i) $K : [0, \infty) \rightarrow (0, a_0^{-1}]$ *and it decreases in ξ .*

(ii) *Type of degeneracy*

$$\frac{c_1}{(1 + \xi)^a} \leq K(\xi) \leq \frac{c_2}{(1 + \xi)^a}. \quad (2.10)$$

(iii) *For all $n \geq 1$,*

$$c_3(\xi^{n-a} - 1) \leq K(\xi)\xi^n \leq c_2\xi^{n-a}. \quad (2.11)$$

(iv) *Relation with its derivative*

$$-aK(\xi) \leq K'(\xi)\xi \leq 0. \quad (2.12)$$

where c_1, c_2, c_3 are positive constants depending on Ω and g and constant

$$a = \frac{\alpha_N}{\alpha_N + 1} = \frac{\deg(g)}{\deg(g) + 1} \in (0, 1).$$

We define

$$H(\xi) = \int_0^{\xi^2} K(\sqrt{s}) dx, \text{ for } \xi \geq 0. \quad (2.13)$$

The function $H(\xi)$ can compare with ξ and $K(\xi)$ by

$$K(\xi)\xi^2 \leq H(\xi) \leq 2K(\xi)\xi^2, \quad (2.14)$$

as a consequence of (2.10)–(2.11)

$$C(\xi^{2-a} - 1) \leq H(\xi) \leq 2C\xi^{2-a}. \quad (2.15)$$

Next we recall important monotonicity properties

LEMMA 2.2 (cf. [15], Lemma 5.2). *For all $y, y' \in \mathbb{R}^d$, one has*

$$(K(|y'|)y' - K(|y|)y) \cdot (y' - y) \geq (1 - a)K(\max\{|y|, |y'|\})|y' - y|^2. \quad (2.16)$$

LEMMA 2.3 (cf. [3], Lemma III.11). *For the vector functions $\mathbf{s}_1, \mathbf{s}_2$, we have*

$$\int_{\Omega} (K(|\mathbf{s}_1|)\mathbf{s}_1 - K(|\mathbf{s}_2|)\mathbf{s}_2) \cdot (\mathbf{s}_1 - \mathbf{s}_2) dx \geq C\omega \|\mathbf{s}_1 - \mathbf{s}_2\|_{L^{2-a}(\Omega)}^2, \quad (2.17)$$

where

$$\omega = (1 + \max\{\|\mathbf{s}_1\|_{L^{2-a}(\Omega)}, \|\mathbf{s}_2\|_{L^{2-a}(\Omega)}\})^{-a}. \quad (2.18)$$

For the continuity of $K(\xi, \vec{a})$ we have the following fact

LEMMA 2.4. *For all $y, y' \in \mathbb{R}^d$. There is a positive constant C such that*

$$|K(|y'|)y' - K(|y|)y| \leq C|y' - y|. \quad (2.19)$$

Proof. Case 1: The origin does not belong to the segment connect y' and y . Let $\ell(t) = ty' + (1 - t)y, t \in [0, 1]$. Define $h(t) = K(|\ell(t)|)\ell(t)$ for $t \in [0, 1]$. By the mean value theorem, there is $t_0 \in [0, 1]$ with $\ell(t_0) \neq 0$, such that

$$\begin{aligned} |K(|y'|)y' - K(|y|)y|^2 &= |h(1) - h(0)|^2 = |h'(t_0)|^2 \\ &= \left| K'(|\ell(t_0)|) \frac{\ell(t_0) \cdot \ell'(t_0)}{|\ell(t_0)|} \ell(t_0) + K(|\ell(t_0)|) \ell'(t_0) \right|^2. \end{aligned}$$

Using (2.12) and Minkowski's inequality we obtain

$$|K(|y'|)y' - K(|y|)y|^2 \leq 2|K(|\ell(t_0)|)|^2 \left\{ a^2 \left| \frac{\ell(t_0) \cdot \ell'(t_0)}{|\ell(t_0)|^2} \ell(t_0) \right|^2 + |\ell'(t_0)|^2 \right\}.$$

The (2.19) follows by the boundedness of $K(\cdot) \leq a_0^{-1}$.

Case 2: The origin belongs to the segment connect y', y . We replace y' by some $y_\varepsilon \neq 0$ so that $0 \notin [y_\varepsilon, y]$ and $y_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Apply the above inequality for y and y_ε , then let $\varepsilon \rightarrow 0$. \square

Notations. Let $L^2(\Omega)$ be the set of square integrable function on Ω and $(L^2(\Omega))^d$ the space of d -dimensional vectors which have all components in $L^2(\Omega)$.

We denote by (\cdot, \cdot) the inner product in either $L^2(\Omega)$ or $(L^2(\Omega))^d$ that is

$$(\xi, \eta) = \int_{\Omega} \xi \eta dx \quad \text{or} \quad (\boldsymbol{\xi}, \boldsymbol{\eta}) = \int_{\Omega} \boldsymbol{\xi} \cdot \boldsymbol{\eta} dx.$$

and $\langle \cdot, \cdot \rangle$ defined by

$$\langle u, v \rangle = \int_{\Gamma} u v d\sigma.$$

The notation $\|\cdot\|$ will means scalar norm $\|\cdot\|_{L^2(\Omega)}$ or vector norm $\|\cdot\|_{(L^2(\Omega))^d}$.

For $1 \leq q \leq +\infty$ and m any nonnegative integer, let

$$W^{m,q}(\Omega) = \{f \in L^q(\Omega), D^\alpha f \in L^q(\Omega), |\alpha| \leq m\}$$

denote a Sobolev space endowed with the norm

$$\|f\|_{m,q} = \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^q(\Omega)}^q \right)^{\frac{1}{q}}.$$

Define $H^m(\Omega) = W^{m,2}(\Omega)$ with the norm $\|\cdot\|_m = \|\cdot\|_{m,2}$.

For functions p, u and vector functions $\mathbf{u}, \mathbf{s}, \mathbf{v}$ we use short hand notations

$$\|p(t)\| = \|p(\cdot, t)\|_{L^2(\Omega)}, \quad \|\mathbf{u}(t)\| = \|\mathbf{u}(\cdot, t)\|_{L^2(\Omega)}, \quad \|\mathbf{s}(t)\|_{L^{2-a}} = \|\mathbf{s}(\cdot, t)\|_{L^{2-a}(\Omega)}$$

and

$$u^0 = u(\cdot, 0), \quad \mathbf{v}^0 = \mathbf{v}(\cdot, 0).$$

for all functions u and vector functions \mathbf{v} .

Throughout this paper the constants

$$\beta = 2 - a, \quad \lambda = \frac{2 - \beta}{\beta}, \quad \delta = \frac{\beta}{\beta - 1}.$$

The arguments C, C_1 will represent for positive generic constants and their values depend on exponents, coefficients of polynomial g , the spatial dimension d and domain Ω , independent of the initial and boundary data, size of mesh and time step. These constants may be different place by place.

3. Expanded mixed finite element methods. In this section, we develop the semidiscrete expanded mixed finite element method for the problem (2.8) and a fully discrete version.

Consider the initial value boundary problem (IVBP):

$$\begin{cases} p_t + \nabla \cdot \mathbf{u} = f, \\ \mathbf{u} + K(|\mathbf{s}|)\mathbf{s} = 0, \\ \mathbf{s} - \nabla p = 0, \end{cases} \quad (3.1)$$

for all $x \in \Omega, t \in (0, T)$, where $f : \Omega \times (0, T) \rightarrow \mathbb{R}$ is given and $f \in C^1([0, T]; L^\infty(\Omega))$. We assume the flux condition on the boundary: $\mathbf{u} \cdot \nu = 0, x \in \Gamma, t \in [0, T]$, where ν is the outward normal vector on Γ . The initial data: $p(x, 0) = p_0(x)$ is given.

Let $W = L^2(\Omega)$, $\tilde{W} = (L^2(\Omega))^d$, and the Hilbert space

$$V = H_0(\text{div}, \Omega) = \{\mathbf{v} \in (L^2(\Omega))^d, \nabla \cdot \mathbf{v} \in L^2(\Omega), \mathbf{v} \cdot \nu = 0 \text{ on } \Gamma\}$$

with the norm defined by $\|\mathbf{v}\|_V^2 = \|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2$.

The variational formulation is defined as the following: Find $(p, \mathbf{s}, \mathbf{u}) : [0, T] \rightarrow W \times \tilde{W} \times V$ such that

$$(p_t, w) + (\nabla \cdot \mathbf{u}, w) = (f, w), \quad \forall w \in W, \quad (3.2a)$$

$$(\mathbf{u}, \mathbf{z}) + (K(|\mathbf{s}|)\mathbf{s}, \mathbf{z}) = 0, \quad \forall \mathbf{z} \in \tilde{W}, \quad (3.2b)$$

$$(\mathbf{s}, \mathbf{v}) + (p, \nabla \cdot \mathbf{v}) = 0 \quad \forall \mathbf{v} \in V, \quad (3.2c)$$

with $p(x, 0) = p_0(x)$, $x \in \Omega$ and $\mathbf{u} \cdot \nu = 0$, $x \in \Gamma$, $t \in [0, T]$.

Let $\{\mathcal{T}_h\}_h$ be a family of quasi-uniform triangulations of Ω with h being the maximum diameter of the element. Let V_h be the Raviart-Thomas-Nédélec spaces [24, 29] of order $r \geq 0$ or Brezzi-Douglas-Marini spaces [4] of index r over each triangulation \mathcal{T}_h , W_h the space of discontinuous piecewise polynomials of degree r over \mathcal{T}_h , \tilde{W}_h the n -dimensional vector space of discontinuous piecewise polynomials of degree r over \mathcal{T}_h . Let $W_h \times \tilde{W}_h \times V_h$ be the mixed element spaces approximating to $W \times \tilde{W} \times V$. The semidiscrete expanded mixed formulation of (3.2) can read as following: Find $(p_h, \mathbf{s}_h, \mathbf{u}_h) : [0, T] \rightarrow W_h \times \tilde{W}_h \times V_h$ such that

$$(p_{h,t}, w_h) + (\nabla \cdot \mathbf{u}_h, w_h) = (f, w_h), \quad \forall w_h \in W_h, \quad (3.3a)$$

$$(\mathbf{u}_h, \mathbf{z}_h) + (K(|\mathbf{s}_h|)\mathbf{s}_h, \mathbf{z}_h) = 0, \quad \forall \mathbf{z}_h \in \tilde{W}_h, \quad (3.3b)$$

$$(\mathbf{s}_h, \mathbf{v}_h) + (p_h, \nabla \cdot \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in V_h, \quad (3.3c)$$

where $p_h(x, 0) = \pi p_0(x)$, $\mathbf{u}_h \cdot \nu = 0$, $x \in \Gamma$, $t \in [0, T]$.

We use the standard L^2 -projection operator $\pi : W \rightarrow W_h$, $\pi : \tilde{W} \rightarrow \tilde{W}_h$ satisfying

$$(\pi w, \nabla \cdot \mathbf{v}_h) = (w, \nabla \cdot \mathbf{v}_h) \quad (3.4)$$

for all $w \in W, \mathbf{v}_h \in V_h$, and

$$(\pi \mathbf{z}, \mathbf{z}_h) = (\mathbf{z}, \mathbf{z}_h) \quad (3.5)$$

for all $\mathbf{z} \in \tilde{W}, \mathbf{z}_h \in \tilde{W}_h$.

Also we use H -div projection $\Pi : V \rightarrow V_h$ defined by

$$(\nabla \cdot \Pi \mathbf{v}, w_h) = (\nabla \cdot \mathbf{v}, w_h) \quad (3.6)$$

for all $w_h \in W_h$.

These projections have well-known approximation properties as in [5, 20]. Below are the standard approximation properties for these projections

(i) There exist positive constant C_1, C_2 such that

$$\|\pi w - w\|_{0,\alpha} \leq C_1 h^m \|w\|_{m,\alpha} \quad \text{and} \quad \|\pi \mathbf{z} - \mathbf{z}\|_{0,\alpha} \leq C_2 h^m \|\mathbf{z}\|_{m,\alpha}, \quad (3.7)$$

for all $w \in W^{m,\alpha}(\Omega)$, $\mathbf{z} \in (W^{m,\alpha}(\Omega))^d$, $0 \leq m \leq r+1$, $1 \leq \alpha \leq \infty$. Here $\|\cdot\|_{m,\alpha}$ denotes a standard norm in Sobolev space $W^{m,\alpha}$. In short hand, when $\alpha = 2$ we write (3.7) as

$$\|\pi w - w\| \leq C_1 h^m \|w\|_m, \quad \text{and} \quad \|\pi \mathbf{z} - \mathbf{z}\| \leq C_2 h^m \|\mathbf{z}\|_m. \quad (3.8)$$

(ii) There exists a positive C_3 such that

$$\|\Pi \mathbf{v} - \mathbf{v}\|_{0,\alpha} \leq C_3 h^m \|\mathbf{v}\|_{m,\alpha} \quad (3.9)$$

for any $\mathbf{v} \in (W^{m,\alpha}(\Omega))^d$, $1/\alpha \leq m \leq r+1$, $1 \leq \alpha \leq \infty$.

Because of the commuting relation between π, Π and the divergence (i.e., that $\nabla \cdot \Pi \mathbf{u} = \pi(\nabla \cdot \mathbf{u})$), we also have the bound

$$\|\nabla \cdot (\Pi \mathbf{v} - \mathbf{v})\|_{0,\alpha} \leq C_1 h^m \|\nabla \cdot \mathbf{v}\|_{m,\alpha}, \quad (3.10)$$

provided $\nabla \cdot \mathbf{v} \in W^{m,\alpha}(\Omega)$ for $1 \leq m \leq r+1$.

Let N be the positive integer, $t_0 = 0 < t_1 < \dots < t_n = T$ be partition interval $[0, T]$ of N sub-intervals, and let $\Delta t = t_n - t_{n-1} = T/N$ be the n -th time step size, $t_n = n\Delta t$ and $\varphi^n = \varphi(\cdot, t_n)$.

The discrete time expanded mixed finite element approximation to (3.2) is defined as follows: Find $(p_h^n, \mathbf{s}_h^n, \mathbf{u}_h^n) \in W_h \times \tilde{W}_h \times V_h$, $n = 1, 2, \dots, N$, such that

$$\left(\frac{p_h^n - p_h^{n-1}}{\Delta t}, w_h \right) + (\nabla \cdot \mathbf{u}_h^n, w_h) = (f^n, w_h), \quad \forall w_h \in W_h, \quad (3.11a)$$

$$(\mathbf{u}_h^n, \mathbf{z}_h) + (K(|\mathbf{s}_h^n|)\mathbf{s}_h^n, \mathbf{z}_h) = 0, \quad \forall \mathbf{z}_h \in \tilde{W}_h, \quad (3.11b)$$

$$(\mathbf{s}_h^n, \mathbf{v}_h) + (p_h^n, \nabla \cdot \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in V_h. \quad (3.11c)$$

The initial approximations are chosen by

$$p_h^0(x) = \pi p_0(x), \quad \mathbf{s}_h^0(x) = \nabla p_h^0(x), \quad \mathbf{u}_h^0(x) = K(|\mathbf{s}_h^0(x)|)\mathbf{s}_h^0(x)$$

for all $x \in \Omega$.

4. Estimates of solutions . Using the theory of monotone operators [22, 31, 34], the authors in [17] proved the global existence of weak solution of equation (3.1). Moreover $p \in C([0, T], L^\alpha(\Omega))$, $\alpha \geq 1$ and $L_{loc}^\beta([0, T], W^{1,\beta}(\Omega))$ and $p_t \in L_{loc}^{\beta'}([0, T], (W^{1,\beta}(\Omega))') \cap L_{loc}^2([0, T], L^2(\Omega))$ provided the initial, boundary data and f sufficiently smooth. For a *priori* estimate, we assume that the weak solution is sufficiently regularities both in x and t variables.

THEOREM 4.1. *Let $(p, \mathbf{s}, \mathbf{u})$ be the solution to the problem (3.2). We have*

(i)

$$\sup_{t \in [0, T]} \|p(t)\| \leq \|p^0\| + \int_0^T \|f(t)\| dt. \quad (4.1)$$

(ii) For any $t \in (0, T)$,

$$\int_0^t \|p_t(t)\|^2 dt + \int_\Omega H(x, t) dx + \|p(t)\|^2 \leq C\mathcal{M}(t), \quad (4.2)$$

where

$$\mathcal{M}(t) = \|\mathbf{s}^0\|_{L^\beta(\Omega)}^\beta + \|p^0\|^2 + 2 \int_0^t \|f(\tau)\|^2 d\tau + t \left(\|p^0\| + \int_0^t \|f(\tau)\| d\tau \right). \quad (4.3)$$

(iii) For any $t \in (0, T)$,

$$\begin{aligned} \|\mathbf{s}(t)\|_{L^\beta(\Omega)}^\beta + \|\mathbf{u}(t)\| &\leq C \left\{ \|p^0\|^2 + \left(\int_0^t \|f(\tau)\| d\tau \right)^2 \right. \\ &\quad \left. + \int_0^t e^{-(t-\tau)} \|f(\tau)\|^2 d\tau + 1 \right\}. \end{aligned} \quad (4.4)$$

Proof. (i) In (3.3), picking up $w = p$, $\mathbf{z} = \mathbf{s}$ and $\mathbf{v} = \mathbf{u}$ we have

$$(p_t, p) + (\nabla \cdot \mathbf{u}, p) = (f, p), \quad (4.5a)$$

$$(\mathbf{u}, \mathbf{s}) + (K(|\mathbf{s}|)\mathbf{s}, \mathbf{s}) = 0, \quad (4.5b)$$

$$(\mathbf{s}, \mathbf{u}) + (p, \nabla \cdot \mathbf{u}) = 0. \quad (4.5c)$$

We add three above equations to obtain

$$\frac{1}{2} \frac{d}{dt} \|p\|^2 + \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\|^2 = (f, p). \quad (4.6)$$

For each $t \in [0, T]$, integrating the previous estimate on $(0, t)$ and taking the supremum in t yield

$$\begin{aligned} \sup_{t \in [0, T]} \|p(t)\|^2 + 2 \int_0^t \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\|^2 d\tau &\leq \|p(0)\|^2 + \int_0^t (f, p) d\tau \\ &\leq \|p(0)\|^2 + \sup_{t \in [0, T]} \|p(t)\| \int_0^t \|f\| d\tau. \end{aligned} \quad (4.7)$$

Dropping the nonnegative term of the left-hand side of (4.7), we have the bound

$$\sup_{t \in [0, T]} \|p(t)\|^2 \leq \|p(0)\|^2 + \sup_{t \in [0, T]} \|p(t)\| \int_0^t \|f\| d\tau.$$

This have the form $x^2 \leq \delta^2 + \eta x$ where

$$\begin{aligned} x &= \sup_{t \in [0, T]} \|p(t)\|^2, \\ \eta &= \int_0^T \|f\| d\tau \geq 0, \\ \delta &= \|p(0)\| \geq 0. \end{aligned}$$

The element quadratic inequality shows that $x \leq \delta + \eta$. Hence it proves (4.1).

(ii) Selecting $w = p_t$, $\mathbf{z} = \mathbf{s}_t$ in (3.3a), (3.3b), differentiating (3.3c) in time and then choosing $\mathbf{v}_h = \mathbf{u}$, we obtain

$$\begin{aligned} (p_t, p_t) + (\nabla \cdot \mathbf{u}, p_t) &= (f, p_t), \\ (\mathbf{u}, \mathbf{s}_t) + (K(|\mathbf{s}|)\mathbf{s}, \mathbf{s}_t) &= 0, \\ (\mathbf{s}_t, \mathbf{u}) + (p_t, \nabla \cdot \mathbf{u}) &= 0. \end{aligned}$$

Summing up three equations gives

$$\|p_t\|^2 + (K(|\mathbf{s}|)\mathbf{s}, \mathbf{s}_t) = (f, p_t). \quad (4.8)$$

Note that the function $H(\cdot)$ in (2.13) gives

$$K(|\mathbf{s}|)\mathbf{s} \cdot \mathbf{s}_t = \frac{1}{2} \frac{d}{dt} H(\mathbf{s}).$$

We rewrite (4.8) as

$$\|p_t\|^2 + \frac{1}{2} \frac{d}{dt} \int_{\Omega} H(x, t) dx = (f, p_t), \quad (4.9)$$

where $H(x, t) = H(\mathbf{s}(x, t))$.

Now adding (4.6) and (4.9) we obtain

$$\|p_t\|^2 + \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} H(x, t) dx + \|p\|^2 \right) + \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\|^2 = (f, p) + (f, p_t). \quad (4.10)$$

Using Cauchy's inequality and the fact that

$$\left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\|^2 = \int_{\Omega} K(|\mathbf{s}|)\mathbf{s}^2 dx \geq \frac{1}{2} \int_{\Omega} H(\mathbf{s}(x, t)) dx.$$

It follows that

$$\|p_t\|^2 + \frac{d}{dt} \left(\int_{\Omega} H(x, t) dx + \|p\|^2 \right) \leq - \int_{\Omega} H(x, t) dx + 2\|f\|^2 + \|p\|^2. \quad (4.11)$$

Integrating above inequality in t , using (4.1), we find that

$$\begin{aligned} \int_0^t \|p_t(\tau)\|^2 d\tau + \int_{\Omega} H(x, t) dx + \|p\|^2 &\leq - \int_0^t \int_{\Omega} H(x, t) dx \\ &+ \int_{\Omega} H(x, 0) dx + \|p(0)\|^2 + 2 \int_0^t \|f\|^2 + t \left(\|p(0)\| + \int_0^T \|f\| dt \right). \end{aligned} \quad (4.12)$$

Dropping the negative term on the right hand side of (4.11) and using the fact that $H(x, 0) \leq C|\mathbf{s}(x, 0)|^\beta$ we obtain (4.2).

(iii) We rewrite equation (4.10) as form

$$\begin{aligned} \|p_t\|^2 + \frac{1}{2} \frac{d}{dt} \int_{\Omega} H(x, t) dx &= - \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\|^2 + (f, p + p_t) - (p, p_t) \\ &\leq - \frac{1}{2} \int_{\Omega} H(x, t) + \frac{1}{2} \left(\|f\|^2 + \|p\|^2 + \|p_t\|^2 \right). \end{aligned}$$

This implies

$$\frac{d}{dt} \int_{\Omega} H(x, t) dx \leq - \int_{\Omega} H(x, t) + \|f\|^2 + \|p\|^2.$$

Applying Gronwall's inequality, we obtain

$$\int_{\Omega} H(x, t) dx \leq -e^{-t} \int_{\Omega} H(x, 0) dx + C \int_0^t e^{-(t-\tau)} (\|f\|^2 + \|p\|^2) d\tau.$$

Dropping the first term of the right hand side, using (4.1), we obtain

$$\begin{aligned} \int_{\Omega} H(x, t) dx &\leq C \int_0^t e^{-(t-\tau)} \left\{ \|f\|^2 + \|p(0)\|^2 + \left(\int_0^T \|f(s)\| ds \right)^2 \right\} d\tau \\ &\leq C \left\{ \|p(0)\|^2 + \left(\int_0^T \|f(t)\| dt \right)^2 + \int_0^t e^{-(t-\tau)} \|f(\tau)\|^2 d\tau \right\}. \end{aligned} \quad (4.13)$$

Note that

$$\int_{\Omega} H(x, t) dx \geq C \int_{\Omega} (|\mathbf{s}|^{\beta} - 1) dx = C(\|\mathbf{s}\|_{L^{\beta}(\Omega)}^{\beta} - 1). \quad (4.14)$$

In addition equation (4.5b) leads to

$$\|\mathbf{u}\| \leq \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\| \leq C \|\mathbf{s}\|_{L^{\beta}(\Omega)}. \quad (4.15)$$

Therefore, (4.4) follows from (4.13), (4.14) and (4.15). The proof is complete. \square

Although solution is considered continuous at $t = 0$ in appropriate Lebesgue or Sobolev space. Its time derivative is not. In the following we prove the time derivative solution is bounded.

THEOREM 4.2. *Let $0 < t_0 < T$. For each $t \in [t_0, T]$, we have*

$$\|p_t(t)\|^2 \leq C t_0^{-1} \mathcal{M}(t_0) + C \left(\mathcal{M}(t) + \int_0^t (\|f_t(\tau)\|^2 d\tau) \right), \quad (4.16)$$

where $\mathcal{M}(\cdot)$ is defined as in (4.3).

Proof. We differentiate (3.2) with respect time t to obtain

$$(p_{tt}, w) + (\nabla \cdot \mathbf{u}_t, w) = (f_t, w), \quad \forall w \in W, \quad (4.17a)$$

$$(\mathbf{u}_t, \mathbf{z}) + (K(|\mathbf{s}|)\mathbf{s}_t, \mathbf{z}) + \left(K'(|\mathbf{s}|) \frac{\mathbf{s} \cdot \mathbf{s}_t}{|\mathbf{s}|} \mathbf{s}, \mathbf{z} \right) = 0, \quad \forall \mathbf{z} \in \tilde{W}, \quad (4.17b)$$

$$(\mathbf{s}_t, \mathbf{v}_h) + (p_t, \nabla \cdot \mathbf{v}) = 0, \quad \forall \mathbf{v} \in V. \quad (4.17c)$$

For each $t \in [t_0, T]$, taking $w = p_t$, $\mathbf{z} = \mathbf{s}_t$ and $\mathbf{v} = \mathbf{u}_t$, summing three resultant equations we obtain

$$\frac{1}{2} \frac{d}{dt} \|p_t\|^2 + \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s}_t \right\|^2 = - \left(K'(|\mathbf{s}|) \frac{\mathbf{s} \cdot \mathbf{s}_t}{|\mathbf{s}|} \mathbf{s}, \mathbf{s}_t \right) + (f_t, p_t). \quad (4.18)$$

Using (2.12) and Cauchy's inequality to bound the right hand-side of (4.18) give

$$\left| - \left(K'(|\mathbf{s}|) \frac{\mathbf{s} \cdot \mathbf{s}_t}{|\mathbf{s}|} \mathbf{s}, \mathbf{s}_t \right) + (f_t, p_t) \right| \leq a \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s}_t \right\|^2 + \frac{1}{2} (\|f_t\|^2 + \|p_t\|^2). \quad (4.19)$$

Thus

$$\frac{1}{2} \frac{d}{dt} \|p_t\|^2 + (1-a) \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s}_t \right\|^2 \leq \frac{1}{2} (\|f_t\|^2 + \|p_t\|^2).$$

Ignoring the the nonnegative term of the left hand side in previous inequality we find that

$$\frac{d}{dt} \|p_t\|^2 \leq \|p_t\|^2 + \|f_t\|^2. \quad (4.20)$$

For $t \geq t' > 0$, integrating (4.20) from t' to t yields

$$\begin{aligned} \|p_t\|^2 &\leq \|p_t(t')\|^2 + \int_{t'}^t \|p_t\|^2 d\tau + \int_0^t \|f_t\|^2 d\tau \\ &\leq \|p_t(t')\|^2 + \int_0^t \|p_t\|^2 d\tau + \int_0^t \|f_t\|^2 d\tau. \end{aligned}$$

Now integrating in t' from 0 to t_0 ,

$$t_0 \|p_t\|^2 \leq \int_0^{t_0} \|p_t(t')\|^2 dt' + t_0 \left\{ \int_0^t \|p_t\|^2 d\tau + \int_0^t \|f_t\|^2 d\tau \right\}. \quad (4.21)$$

Combining (4.21) and (4.2) leads to (4.24). The proof is complete. \square

Using L^2 -projection, H -div projection and above arguments, the similar results for solution of discrete problem are established as following.

THEOREM 4.3. *Let $(p_h, \mathbf{s}_h, \mathbf{u}_h)$ be the solution to the semidiscrete problem (3.3). We have*

(i)

$$\sup_{t \in [0, T]} \|p_h(t)\|^2 \leq \|p^0\|^2 + \int_0^T \|f(t)\|^2 dt. \quad (4.22)$$

(ii) For any $t \in (0, T)$,

$$\begin{aligned} \|\mathbf{s}_h(t)\|_{L^\beta(\Omega)}^\beta + \|\mathbf{u}_h(t)\| &\leq C \left\{ \|p^0\|^2 + \left(\int_0^T \|f(t)\|^2 dt \right)^2 \right. \\ &\quad \left. + \int_0^t e^{-(t-\tau)} \|f(\tau)\|^2 d\tau + 1 \right\}. \end{aligned} \quad (4.23)$$

(iii) Let $0 < t_0 < T$. For any $t \in [t_0, T]$, we have

$$\|p_{h,t}(t)\|^2 \leq C t_0^{-1} \mathcal{M}(t_0) + C \left(\mathcal{M}(t) + \int_0^t (\|f_t(\tau)\|^2 d\tau) \right), \quad (4.24)$$

where $\mathcal{M}(\cdot)$ is defined as in (4.3).

5. Error analysis. In this section, we will establish the error estimates between the analytical solution and approximation solution in several norms. In the below development we discuss error estimates for the case conductivity tensor $K(\cdot)$ degenerating. We assume the solutions,

$$p \in L^\infty(0, T; H^{r+1}(\Omega)), \quad \mathbf{s} \in L^2(0, T; (W^{r+1, \beta}(\Omega))^d).$$

5.1. Error estimate for semidiscrete method. We find the error bounds in the semidiscrete method by comparing the computed solution to the projections of the true solutions. To do this, we restrict the test functions in (3.2) to the finite dimensional spaces. Let

$$\begin{aligned} p_h - p &= (p_h - \pi p) + (\pi p - p) \equiv \vartheta + \theta, \\ \mathbf{s}_h - \mathbf{s} &= (\mathbf{s}_h - \pi \mathbf{s}) + (\pi \mathbf{s} - \mathbf{s}) \equiv \eta + \zeta, \\ \mathbf{u}_h - \mathbf{u} &= (\mathbf{u}_h - \Pi \mathbf{u}) + (\Pi \mathbf{u} - \mathbf{u}) \equiv \rho + \varrho. \end{aligned}$$

Properties of projections in (3.7) and (3.9) yield

$$\|\theta\|_{L^\alpha(\Omega)} \leq Ch^m \|p\|_{m,\alpha}, \quad \forall p \in W^{m,\alpha}(\Omega), \quad (5.1)$$

$$\|\zeta\|_{L^\alpha(\Omega)} \leq Ch^m \|\mathbf{s}\|_{m,\alpha}, \quad \forall \mathbf{s} \in (W^{m,\alpha}(\Omega))^d, \quad (5.2)$$

$$\|\varrho\|_{L^\alpha(\Omega)} \leq Ch^m \|\mathbf{u}\|_{m,\alpha}, \quad \forall \mathbf{u} \in (W^{m,\alpha}(\Omega))^d. \quad (5.3)$$

for all $1 \leq m \leq r+1$, $1 \leq \alpha \leq \infty$.

Let $0 < t_0 < T$,

$$\begin{aligned} \mathcal{A} &= 1 + \|p^0\|^2 + \left(\int_0^T \|f(t)\| dt \right)^2 + \int_0^T \|f(t)\|^2 dt. \\ \mathcal{B} &= Ct_0^{-1} \mathcal{M}(t_0) + C \left(\mathcal{M}(T) + \int_0^T \|f_t(t)\|^2 dt \right). \end{aligned}$$

THEOREM 5.1. *Assume $(p^0, \mathbf{u}^0, \mathbf{s}^0) \in W \times V \times \tilde{W}$ and $(p_h^0, \mathbf{u}_h^0, \mathbf{s}_h^0) \in W_h \times V_h \times \tilde{W}_h$. Let $(p, \mathbf{u}, \mathbf{s})$ solve problem (3.2) and $(p_h, \mathbf{u}_h, \mathbf{s}_h)$ solve the semidiscrete mixed finite element approximation (3.3). Then there is a positive constant C such that for each $t \in (0, T)$*

$$\|(p_h - p)(t)\| \leq Ch^{r+1} \|p(t)\| + C \mathcal{A}^{\frac{1}{2}} h^{\frac{r+1}{2}} \sqrt{\int_0^t \|\mathbf{s}(\tau)\|_{L^\beta(\Omega)}^2 d\tau}. \quad (5.4)$$

Furthermore if $\mathbf{s} \in L^2(0, T; (W^{r+1,\delta}(\Omega))^d)$ then

$$\|(p_h - p)(t)\| \leq Ch^{r+1} \|p(t)\| + C \mathcal{A}^{\frac{\lambda}{2}} h^{r+1} \sqrt{\int_0^t \|\mathbf{s}(\tau)\|_{L^\delta(\Omega)}^2 d\tau}. \quad (5.5)$$

Proof. Subtracting (3.3) from (3.2) we have the following error equations

$$(p_{h,t} - p_t, w_h) + (\nabla \cdot (\mathbf{u}_h - \mathbf{u}), w_h) = 0, \quad \forall w_h \in W_h, \quad (5.6a)$$

$$(\mathbf{u}_h - \mathbf{u}, \mathbf{z}_h) + (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|)\mathbf{s}, \mathbf{z}_h) = 0, \quad \forall \mathbf{z}_h \in \tilde{W}_h, \quad (5.6b)$$

$$(\mathbf{s}_h - \mathbf{s}, \mathbf{v}_h) + (p_h - p, \nabla \cdot \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in V_h. \quad (5.6c)$$

Let take $w_h = \vartheta$, $\mathbf{z}_h = \eta$ and $\mathbf{v}_h = \rho$. Using the projections in (3.7) and (3.9), we rewrite (5.6) as

$$(\vartheta_t, \vartheta) + (\nabla \cdot \rho, \vartheta) = 0, \quad (5.7a)$$

$$(\rho, \eta) + (K(|\mathbf{s}|)\mathbf{s} - K(|\mathbf{s}_h|)\mathbf{s}_h, \eta) = 0, \quad (5.7b)$$

$$(\eta, \rho) + (\vartheta, \nabla \cdot \rho) = 0. \quad (5.7c)$$

Summing up three equations (5.7a)–(5.7c) gives

$$\frac{1}{2} \frac{d}{dt} \|\vartheta\|^2 + (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|)\mathbf{s}, \eta) = 0.$$

It is equivalent to

$$\frac{1}{2} \frac{d}{dt} \|\vartheta\|^2 + (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|)\mathbf{s}, \mathbf{s}_h - \mathbf{s}) = (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|)\mathbf{s}, \zeta). \quad (5.8)$$

Applying (2.17) to the second term of (5.8) we have

$$(K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|\mathbf{s}), \mathbf{s}_h - \mathbf{s}) \geq C\omega \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 \quad (5.9)$$

with

$$\omega = \omega(t) = (1 + \max\{\|\mathbf{s}_h(t)\|_{L^\beta(\Omega)}, \|\mathbf{s}(t)\|_{L^\beta(\Omega)}\})^{-a}. \quad (5.10)$$

Since $K(|\xi|)\xi \leq C\xi^{\beta-1}$, the right hand side of (5.8) is bounded by

$$\begin{aligned} |(K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|\mathbf{s}), \zeta)| &\leq C(|\mathbf{s}_h|^{\beta-1} + |\mathbf{s}|^{\beta-1}, |\zeta|) \\ &\leq C\left(\|\mathbf{s}_h\|_{L^\beta(\Omega)}^{\beta-1} + \|\mathbf{s}\|_{L^\beta(\Omega)}^{\beta-1}\right) \|\zeta\|_{L^\beta(\Omega)} \\ &\leq C\left(1 + \|\mathbf{s}_h\|_{L^\beta(\Omega)}^\beta + \|\mathbf{s}\|_{L^\beta(\Omega)}^\beta\right) \|\zeta\|_{L^\beta(\Omega)}. \end{aligned}$$

Due to (4.4) and (4.23),

$$\begin{aligned} 1 + \|\mathbf{s}_h\|_{L^\beta(\Omega)}^\beta + \|\mathbf{s}\|_{L^\beta(\Omega)}^\beta &\leq C\left[1 + \|p^0\|^2 + \left(\int_0^T \|f\| dt\right)^2 + \int_0^t e^{-(t-\tau)} \|f\|^2 d\tau\right] \\ &\leq C\mathcal{A}. \end{aligned} \quad (5.11)$$

Hence

$$|(K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|\mathbf{s}), \zeta)| \leq C\mathcal{A} \|\zeta\|_{L^\beta(\Omega)} \quad (5.12)$$

Combining (5.8), (5.9) and (5.12) leads to

$$\frac{d}{dt} \|\vartheta\|^2 + \omega \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 \leq C\mathcal{A} \|\zeta\|_{L^\beta(\Omega)}. \quad (5.13)$$

Integrating (5.13) in time, using $\vartheta(0) = 0$, we have

$$\|\vartheta\|^2 + \int_0^t \omega \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 d\tau \leq C\mathcal{A} \int_0^t \|\zeta\|_{L^\beta(\Omega)} d\tau. \quad (5.14)$$

Ignoring the second term of (5.14) and using the triangle inequality $\|p_h - p\| \leq \|\vartheta\| + \|\theta\|$ we obtain

$$\|p_h - p\|^2 \leq \|\theta\|^2 + C\mathcal{A} \int_0^t \|\zeta\|_{L^\beta(\Omega)} d\tau, \quad (5.15)$$

which proves (5.4).

Under the assumption more on the regularity of solution we bound the right hand side of (5.8) using (2.19), Hölder and Young's inequality to obtain

$$\begin{aligned} |(K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|\mathbf{s}), \zeta)| &\leq C(|\mathbf{s}_h - \mathbf{s}|, |\zeta|) \\ &\leq C \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)} \|\zeta\|_{L^\delta(\Omega)} \\ &\leq \varepsilon \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 + C\varepsilon^{-1} \|\zeta\|_{L^\delta(\Omega)}^2 \end{aligned} \quad (5.16)$$

for all $\varepsilon > 0$.

From (5.8), (5.9) and (5.16), we find that

$$\frac{d}{dt} \|\vartheta\|^2 + C\omega \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 \leq \varepsilon \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 + C\varepsilon^{-1} \|\zeta\|_{L^\delta(\Omega)}^2.$$

Due to (5.10) and (5.11),

$$\omega^{-1} \leq C_1 \left(1 + \|\mathbf{s}\|_{L^\beta(\Omega)}^\beta + \|\mathbf{s}_h\|_{L^\beta(\Omega)}^\beta \right)^\lambda \leq C_1 \mathcal{A}^\lambda. \quad (5.17)$$

Thus

$$\frac{d}{dt} \|\vartheta\|^2 + C(C_1 \mathcal{A}^\lambda)^{-1} \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 \leq \varepsilon \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 + C\varepsilon^{-1} \|\zeta\|_{L^\delta(\Omega)}^2. \quad (5.18)$$

Selecting $\varepsilon = \frac{C}{2C_1 \mathcal{A}^\lambda}$, integrating (5.18) in time, we have

$$\|\vartheta\|^2 + (C\mathcal{A}^\lambda)^{-1} \int_0^t \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 d\tau \leq C\mathcal{A}^\lambda \int_0^t \|\zeta\|_{L^\delta(\Omega)}^2 d\tau. \quad (5.19)$$

Dropping the second term in (5.19) and using triangle inequality $\|p_h - p\| \leq \|\theta\| + \|\vartheta\|$ in (5.19) shows that

$$\|p_h - p\|^2 \leq C \left(\|\theta\|^2 + \mathcal{A}^\lambda \int_0^t \|\zeta\|_{L^\delta(\Omega)}^2 d\tau \right).$$

This together (5.2) and (5.3) gives (5.5). The proof is complete. \square

The L^2 -error estimate and the inverse estimate enable us to have the L^∞ - error estimate as the following

THEOREM 5.2. *Assume $(p^0, \mathbf{u}^0, \mathbf{s}^0) \in W \times V \times \tilde{W}$ and $(p_h^0, \mathbf{u}_h^0, \mathbf{s}_h^0) \in W_h \times V_h \times \tilde{W}_h$. Let $(p, \mathbf{u}, \mathbf{s})$ solve problem (3.2) and $(p_h, \mathbf{u}_h, \mathbf{s}_h)$ solve the semidiscrete mixed finite element approximation (3.3). If $p \in L^\infty(0, T, W^{r+1, \infty}(\Omega))$ then there exists a positive constant C such that for each $t \in (0, T)$,*

$$\|(p - p_h)(t)\|_{L^\infty(\Omega)} \leq Ch^{r+1} \|p(t)\|_{r+1, \infty} + C\mathcal{A}^{\frac{1}{2}} h^{\frac{r-1}{2}} \sqrt{\int_0^t \|\mathbf{s}(t)\|_{r+1, \beta}^2}. \quad (5.20)$$

Furthermore if $\mathbf{s} \in L^2(0, T; (W^{r+1, \delta}(\Omega))^d)$ then

$$\|(p - p_h)(t)\|_{L^\infty(\Omega)} \leq Ch^{r+1} \|p(t)\|_{r+1, \infty} + C\mathcal{A}^{\frac{1}{2}} h^r \sqrt{\int_0^t \|\mathbf{s}(t)\|_{r+1, \delta}^2}. \quad (5.21)$$

Proof. For quasi-uniformly of \mathcal{T}_h , the following inverse estimate holds

$$\|\vartheta\|_{L^\infty(\Omega)} \leq Ch^{-\frac{2}{q}} \|\vartheta\|_{L^q(\Omega)} \quad \text{for all } 1 \leq q \leq \infty.$$

Applying this with $q = 2$ and using (5.14) imply

$$\|\vartheta\|_{L^\infty(\Omega)} \leq Ch^{-1} \|\vartheta\| \leq C\mathcal{A}^{\frac{1}{2}} h^{-1} \left(\int_0^t \|\zeta\|_{L^\beta(\Omega)}^2 \right)^{\frac{1}{2}}. \quad (5.22)$$

It follows from triangle inequality and (5.22) that

$$\begin{aligned} \|p - p_h\|_{L^\infty(\Omega)} &\leq \|\theta\|_{L^\infty(\Omega)} + \|\vartheta\|_{L^\infty(\Omega)} \\ &\leq \|\theta\|_{L^\infty(\Omega)} + C\mathcal{A}^{\frac{1}{2}}h^{-1} \left(\int_0^t \|\zeta\|_{L^\beta(\Omega)} \right)^{\frac{1}{2}}. \end{aligned} \quad (5.23)$$

Thus (5.20) follows by (5.23) and (5.2) applying with $\alpha = \infty$.

Using (5.19) to bound $\|\vartheta\|_{L^\infty(\Omega)}$ instead of (5.14) we obtain

$$\|p - p_h\|_{L^\infty(\Omega)} \leq \|\theta\|_{L^\infty(\Omega)} + C\mathcal{A}^{\frac{1}{2}}h^{-1} \left(\int_0^t \|\zeta\|_{L^\delta(\Omega)}^2 \right)^{\frac{1}{2}}. \quad (5.24)$$

This and (5.2) applying with $\alpha = \infty$ give (5.24). We finish the proof. \square

Return to error estimate for vector gradient of pressure we have the following results

THEOREM 5.3. *Under the assumptions of Theorem 5.1. For any $0 < t_0 \leq t \leq T$ there is positive constants C independent of h such that*

(i)

$$\begin{aligned} \|(\mathbf{s}_h - \mathbf{s})(t)\|_{L^\beta(\Omega)} &\leq C\mathcal{A}^{\frac{2\lambda+1}{4}}\mathcal{B}^{\frac{1}{2}}h^{\frac{r+1}{4}} \sqrt{\int_0^t \|\mathbf{s}(\tau)\|_{r+1,\beta} d\tau} \\ &\quad + C\mathcal{A}^{\frac{\lambda+1}{2}}h^{\frac{r+1}{2}} \|\mathbf{s}(t)\|_{r+1,\beta}. \end{aligned} \quad (5.25)$$

and

$$\begin{aligned} \|(\mathbf{u}_h - \mathbf{u})(t)\|_{L^\beta(\Omega)} &\leq C\mathcal{A}^{\frac{2\lambda+1}{4}}\mathcal{B}^{\frac{1}{2}}h^{\frac{r+1}{4}} \sqrt{\int_0^t \|\mathbf{s}(\tau)\|_{r+1,\beta} d\tau} \\ &\quad + C\mathcal{A}^{\frac{\lambda+1}{2}}h^{\frac{r+1}{2}} \|\mathbf{s}(t)\|_{r+1,\beta} + Ch^{r+1} \|\mathbf{u}(t)\|_{r+1,\beta}. \end{aligned} \quad (5.26)$$

(ii) If $\mathbf{s} \in L^2(0, T; (W^{r+1,\delta}(\Omega))^d)$ then

$$\begin{aligned} \|(\mathbf{s}_h - \mathbf{s})(t)\|_{L^\beta(\Omega)} &\leq C\mathcal{A}^{\frac{3\lambda}{4}}\mathcal{B}^{\frac{1}{2}}h^{\frac{r+1}{2}} \sqrt{\int_0^t \|\mathbf{s}(\tau)\|_{r+1,\lambda}^2 d\tau} \\ &\quad + C\mathcal{A}h^{r+1} \|\mathbf{s}(t)\|_{r+1,\delta}. \end{aligned} \quad (5.27)$$

and

$$\begin{aligned} \|(\mathbf{u}_h - \mathbf{u})(t)\|_{L^\beta(\Omega)} &\leq C\mathcal{A}^{\frac{3\lambda}{4}}\mathcal{B}^{\frac{1}{2}}h^{\frac{r+1}{2}} \sqrt{\int_0^t \|\mathbf{s}(\tau)\|_{r+1,\delta}^2 d\tau} \\ &\quad + C\mathcal{A}h^{r+1} \|\mathbf{s}(t)\|_{r+1,\delta} + Ch^{r+1} \|\mathbf{u}(t)\|_{r+1,\beta}. \end{aligned} \quad (5.28)$$

Proof.

(i) Thank to (5.9), (5.8) and L^2 -projection,

$$\begin{aligned} \omega \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 &\leq (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|\mathbf{s}), \mathbf{s}_h - \mathbf{s}) \\ &= -(p_{h,t} - p_t, \vartheta) + (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|\mathbf{s}), \zeta). \end{aligned} \quad (5.29)$$

This, (5.12) and (5.14) yield

$$\begin{aligned} \omega \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 &\leq C(\|p_{h,t}\| + \|p_t\|) \|\vartheta\| + C\mathcal{A} \|\zeta\|_{L^\beta(\Omega)} \\ &\leq C\mathcal{A}^{\frac{1}{2}}\mathcal{B} \left(\int_0^t \|\zeta\|_{L^\beta(\Omega)} d\tau \right)^{\frac{1}{2}} + C\mathcal{A} \|\zeta\|_{L^\beta(\Omega)}. \end{aligned} \quad (5.30)$$

Thus

$$\|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 \leq C\mathcal{A}^{\frac{1}{2}}\mathcal{B}\omega^{-1} \left(\int_0^t \|\zeta\|_{L^\beta(\Omega)} d\tau \right)^{\frac{1}{2}} + C\omega^{-1}\mathcal{A} \|\zeta\|_{L^\beta(\Omega)}. \quad (5.31)$$

Due to (5.17), we obtain

$$\|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 \leq C\mathcal{A}^{\lambda+\frac{1}{2}}\mathcal{B} \left(\int_0^t \|\zeta(\tau)\|_{L^\beta(\Omega)} d\tau \right)^{\frac{1}{2}} + C\mathcal{A}^{\lambda+1} \|\zeta(t)\|_{L^\beta(\Omega)}.$$

Hence (5.25) follows by (5.2).

In (5.6b), let $\mathbf{z}_h = \rho^{\beta-1} \in \tilde{W}_h$ and use Hölder's inequality we obtain

$$\begin{aligned} \|\rho\|_{L^\beta(\Omega)}^\beta &= -(K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|)\mathbf{s}, \rho^{\beta-1}) \\ &\leq C(|\mathbf{s}_h - \mathbf{s}|, \rho^{\beta-1}) \\ &\leq C\|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)} \|\rho\|_{L^\beta(\Omega)}^{\beta-1}, \end{aligned} \quad (5.32)$$

which gives

$$\|\rho\|_{L^\beta(\Omega)} \leq C\|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}.$$

Hence

$$\begin{aligned} \|\mathbf{u}_h - \mathbf{u}\|_{L^\beta(\Omega)} &\leq C \left(\|\rho\|_{L^\beta(\Omega)} + \|\varrho\|_{L^\beta(\Omega)} \right) \\ &\leq C \left(\|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)} + \|\varrho\|_{L^\beta(\Omega)} \right). \end{aligned} \quad (5.33)$$

Using (5.25) and (5.3) we obtain (5.26).

(ii) We bound the right hand side of (5.29) by using Cauchy- Schwartz, triangle inequality and (5.16) to obtain

$$\begin{aligned} \omega \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 &\leq C(\|p_{h,t}\| + \|p_t\|) \|\vartheta\| + \varepsilon \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 + C\varepsilon^{-1} \|\zeta\|_{L^\delta(\Omega)}^2 \\ &\leq C\mathcal{A}^{\frac{1}{2}}\mathcal{B} \left(\int_0^t \|\zeta\|_{L^\delta(\Omega)}^2 d\tau \right)^{\frac{1}{2}} + \varepsilon \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 + C\varepsilon^{-1} \|\zeta\|_{L^\delta(\Omega)}^2. \end{aligned}$$

Then by (5.17),

$$\begin{aligned} (C_1\mathcal{A}^\lambda)^{-1} \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 &\leq C\mathcal{A}^{\frac{1}{2}}\mathcal{B} \left(\int_0^t \|\zeta\|_{L^\delta(\Omega)}^2 d\tau \right)^{\frac{1}{2}} \\ &\quad + \varepsilon \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 + C\varepsilon^{-1} \|\zeta\|_{L^\delta(\Omega)}^2. \end{aligned}$$

Selecting $\varepsilon = \frac{1}{2C_1\mathcal{A}^\lambda}$ then

$$\|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 \leq C\mathcal{A}^{\frac{3\lambda}{2}}\mathcal{B} \left(\int_0^t \|\zeta\|_{L^\delta(\Omega)}^2 d\tau \right)^{\frac{1}{2}} + C\mathcal{A}^{2\lambda} \|\zeta\|_{L^\delta(\Omega)}^2. \quad (5.34)$$

This and (5.2) lead to (5.27).

Inequality (5.28) follows from (5.33) and (5.27). The proof is complete. \square

5.2. Error analysis for fully discrete scheme. In analyzing this method, proceed in a similar fashion as for the semidiscrete method, we derive a error estimate for the fully discrete scheme. Let $p^n(\cdot) = p(\cdot, t_n)$, $\mathbf{v}^n(\cdot) = \mathbf{v}(\cdot, t_n)$ and $\mathbf{u}^n(\cdot) = \mathbf{u}(\cdot, t_n)$ be the true solution evaluated at the discrete time levels. We will also denote $\pi p^n \in W_h$, $\pi \mathbf{s}^n \in \tilde{W}_h$ and $\Pi \mathbf{u}^n \in V_h$ to be the projections of the true solutions at the discrete time levels.

We rewrite (3.2) with $t = t_n$. Using the definitions of projections and assumption that $\nabla \cdot V_h \subset W_h$, standard manipulations show that the true solution satisfies the discrete equation

$$\left(\frac{\pi p^n - \pi p^{n-1}}{\Delta t}, w_h \right) + (\nabla \cdot \Pi \mathbf{u}^n, w_h) = (f^n, w_h) + (\epsilon^n, w_h), \quad \forall w_h \in W_h \quad (5.35a)$$

$$(\Pi \mathbf{u}^n, \mathbf{z}_h) + (K(|\mathbf{s}^n|) \mathbf{s}^n, \mathbf{z}_h) = 0, \quad \forall \mathbf{z}_h \in \tilde{W}_h, \quad (5.35b)$$

$$(\pi \mathbf{s}^n, \mathbf{v}_h) + (\pi p^n, \nabla \cdot \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in V_h, \quad (5.35c)$$

where ϵ^n is the time truncation error of order Δt .

THEOREM 5.4. *Assume $(\bar{p}^0, \mathbf{u}^0, \mathbf{s}^0) \in W \times V \times \tilde{W}$ and $(\bar{p}_h^0, \mathbf{u}_h^0, \mathbf{s}_h^0) \in W_h \times V_h \times \tilde{W}_h$. Let $(p, \mathbf{u}, \mathbf{s})$ solve problem (3.2) and $(p_h^n, \mathbf{u}_h^n, \mathbf{s}_h^n)$ solve the fully discrete mixed finite element approximation (3.11) for each time step n , $n = 1, \dots, N$. There exists a positive constant C independent of h and Δt such that if the Δt is sufficiently small then*

$$\|p_h^m - p^m\| \leq C(h^{\frac{r+1}{2}} + \Delta t) \quad (5.36)$$

for $m = 1, \dots, N$.

Moreover if $\mathbf{s}^n \in (W^{r+1, \delta}(\Omega))^d$ for $n = 1, \dots, N$ then

$$\|p_h^m - p^m\| \leq C(h^{r+1} + \Delta t) \quad (5.37)$$

for $m = 1, \dots, N$.

Proof. Subtracting (3.11) from (5.35), in the resultants using $w_h = \vartheta^n, \mathbf{z}_h = \eta^n, \mathbf{v}_h = \rho^n$ we obtain

$$\left(\frac{\vartheta^n - \vartheta^{n-1}}{\Delta t}, \vartheta^n \right) + (\nabla \cdot \rho^n, \vartheta^n) = (\epsilon^n, \vartheta^n), \quad (5.38a)$$

$$(\rho^n, \eta^n) + (K(|\mathbf{s}_h^n|) \mathbf{s}_h^n - K(|\mathbf{s}^n|) \mathbf{s}^n, \eta^n) = 0, \quad (5.38b)$$

$$(\eta^n, \rho^n) + (\vartheta^n, \nabla \cdot \rho^n) = 0. \quad (5.38c)$$

Combining (5.38a)–(5.38c) gives

$$\|\vartheta^n\|^2 + \Delta t (K(|\mathbf{s}_h^n|) \mathbf{s}_h^n - K(|\mathbf{s}^n|) \mathbf{s}^n, \eta^n) = (\vartheta^n, \vartheta^{n-1}) + \Delta t (\epsilon^n, \vartheta^n).$$

This equation is equivalent to

$$\begin{aligned} & \|\vartheta^n\|^2 + \Delta t (K(|\mathbf{s}_h^n|) \mathbf{s}_h^n - K(|\mathbf{s}^n|) \mathbf{s}^n, \mathbf{s}_h^n - \mathbf{s}^n) \\ &= (\vartheta^n, \vartheta^{n-1}) + \Delta t \left\{ (K(|\mathbf{s}_h^n|) \mathbf{s}_h^n - K(|\mathbf{s}^n|) \mathbf{s}^n, \zeta^n) + (\epsilon^n, \vartheta^n) \right\}. \end{aligned} \quad (5.39)$$

The second term of (5.39), using (2.16), is bounded :

$$(K(|\mathbf{s}_h^n|) \mathbf{s}_h^n - K(|\mathbf{s}^n|) \mathbf{s}^n, \mathbf{s}_h^n - \mathbf{s}^n) \geq C \omega^n \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2, \quad (5.40)$$

where $\omega^n = \omega(t_n)$.

The right hand side of (5.39) using Cauchy's inequality and (5.12) and (5.11) give

$$\begin{aligned} & (\vartheta^n, \vartheta^{n-1}) + \Delta t ((K(|\mathbf{s}_h^n|)\mathbf{s}_h^n - K(|\mathbf{s}^n|)\mathbf{s}^n, \zeta^n) + (\epsilon^n, \vartheta^n)) \\ & \leq \frac{1}{2} (\|\vartheta^n\|^2 + \|\vartheta^{n-1}\|^2) + \Delta t \left\{ C\mathcal{A} \|\zeta^n\|_{L^\beta(\Omega)} + \frac{1}{2} (\|\vartheta^n\|^2 + \|\epsilon^n\|^2) \right\}. \end{aligned} \quad (5.41)$$

It follows from (5.39), (5.40) and (5.41) that

$$\|\vartheta^n\|^2 - \|\vartheta^{n-1}\|^2 + C\Delta t \omega^n \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 \leq \Delta t \|\vartheta^n\|^2 + C\Delta t (\mathcal{A} \|\zeta^n\|_{L^\beta(\Omega)} + \|\epsilon^n\|^2).$$

Summing over n

$$\begin{aligned} & (1 - \Delta t) \|\vartheta^m\|^2 + C \sum_{n=1}^m \Delta t \omega^n \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 \\ & \leq \sum_{n=1}^{m-1} \Delta t \|\vartheta^n\|^2 + C \sum_{n=1}^m \Delta t (\mathcal{A} \|\zeta^n\|_{L^\beta(\Omega)} + \|\epsilon^n\|^2) \end{aligned}$$

for some $m = 2, \dots, N$.

Dropping the nonnegative term of the left hand side, using Gronwall's lemma, we obtain

$$\|\vartheta^m\|^2 \leq C \sum_{n=1}^m \Delta t (\mathcal{A} \|\zeta^n\|_{L^\beta(\Omega)} + \|\epsilon^n\|^2). \quad (5.42)$$

The triangle inequality gives

$$\|p_h^m - p^m\|^2 \leq C\mathcal{A} \sum_{n=1}^m \Delta t \|\zeta^n\|_{L^\beta(\Omega)} + \|\vartheta^m\|^2 + C(\Delta t)^2.$$

This and properties of projections lead to (5.36) true.

(ii) We prove the superconvergence by estimate the right hand side of (5.39) using Cauchy's inequality, (5.16) and (5.11) to obtain

$$\begin{aligned} & (\vartheta^n, \vartheta^{n-1}) + \Delta t ((K(|\mathbf{s}_h^n|)\mathbf{s}_h^n - K(|\mathbf{s}^n|)\mathbf{s}^n, \zeta^n) + (\epsilon^n, \vartheta^n)) \leq \frac{1}{2} (\|\vartheta^n\|^2 + \|\vartheta^{n-1}\|^2) \\ & + \Delta t \left\{ \varepsilon \omega^n \|\mathbf{s}^n - \mathbf{s}_h^n\|_{L^\beta(\Omega)}^2 + C_1 (\varepsilon \omega^n)^{-1} \|\zeta^n\|_{L^\delta(\Omega)}^2 + \frac{1}{2} (\|\vartheta^n\|^2 + \|\epsilon^n\|^2) \right\}. \end{aligned} \quad (5.43)$$

Now we combine (5.40), (5.39) and (5.43) to have

$$\begin{aligned} \|\vartheta^n\|^2 - \|\vartheta^{n-1}\|^2 + C\Delta t \omega^n \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 & \leq \Delta t \|\vartheta^n\|^2 + 2\varepsilon \Delta t \omega^n \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 \\ & + C_1 \Delta t ((\varepsilon \omega^n)^{-1} \|\zeta^n\|_{L^\delta(\Omega)}^2 + \|\epsilon^n\|^2). \end{aligned}$$

Selecting $\varepsilon = C/4$ we obtain

$$\begin{aligned} \|\vartheta^n\|^2 - \|\vartheta^{n-1}\|^2 + C\Delta t \omega^n \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 & \leq \Delta t \|\vartheta^n\|^2 + C_2 \Delta t ((\omega^n)^{-1} \|\zeta^n\|_{L^\delta(\Omega)}^2 + \|\epsilon^n\|^2) \\ & \leq \Delta t \|\vartheta^n\|^2 + C_2 \Delta t (\mathcal{A}^\lambda \|\zeta^n\|_{L^\delta(\Omega)}^2 + \|\epsilon^n\|^2). \end{aligned}$$

Now we drop the the nonnegative term in the left hand side in above inequality, sum over n and use Gronwall's inequality to find that

$$\|\vartheta^m\|^2 \leq C \sum_{n=1}^m \Delta t \left(\mathcal{A}^\lambda \|\zeta^n\|_{L^\delta(\Omega)}^2 + \|\epsilon^n\|^2 \right).$$

Again using triangle inequality, properties of projections we obtain (5.37). \square

THEOREM 5.5. *Under the assumptions of Theorem 5.4. There exists a positive constant C independent of h and Δt such that if the Δt is sufficiently small then*

$$\|\mathbf{s}_h^m - \mathbf{s}^m\|_{L^\beta(\Omega)} + \|\mathbf{u}_h^m - \mathbf{u}^m\|_{L^\beta(\Omega)} \leq C(h^{\frac{r+1}{4}} + \sqrt{\Delta t}) \quad (5.44)$$

for all $m = 1, \dots, N$.

Furthermore if $\mathbf{s}^n \in (W^{r+1,\delta}(\Omega))^d$ for all $n = 1, \dots, N$ then

$$\|\mathbf{s}_h^m - \mathbf{s}^m\|_{L^\beta(\Omega)} + \|\mathbf{u}_h^m - \mathbf{u}^m\|_{L^\beta(\Omega)} \leq C(h^{\frac{r+1}{2}} + \sqrt{\Delta t}) \quad (5.45)$$

for all $m = 1, \dots, N$.

Proof. Recall that the true solution satisfies the discrete equations

$$(p_t^n, w_h) + (\nabla \cdot \Pi \mathbf{u}^n, w_h) = (f^n, w_h), \quad \forall w_h \in W_h \quad (5.46a)$$

$$(\Pi \mathbf{u}^n, \mathbf{z}_h) + (K(|\mathbf{s}^n|) \mathbf{s}^n, \mathbf{z}_h) = 0, \quad \forall \mathbf{z}_h \in \tilde{W}_h, \quad (5.46b)$$

$$(\pi \mathbf{s}^n, \mathbf{v}_h) + (\pi p^n, \nabla \cdot \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in V_h, \quad (5.46c)$$

Subtracting (3.11) from (5.46), choosing $w_h = \vartheta^n$, $\mathbf{z}_h = \eta^n$, $\mathbf{v}_h = \rho^n$, we obtain

$$\left(\frac{p_h^n - p_h^{n-1}}{\Delta t} - p_t^n, \vartheta^n \right) + (\nabla \cdot \rho^n, \vartheta^n) = 0, \quad (5.47a)$$

$$(\rho^n, \eta^n) + (K(|\mathbf{s}_h^n|) \mathbf{s}_h^n - K(|\mathbf{s}^n|) \mathbf{s}^n, \eta^n) = 0, \quad (5.47b)$$

$$(\eta^n, \rho^n) + (\vartheta^n, \nabla \cdot \rho^n) = 0. \quad (5.47c)$$

Above equations yield

$$\left(\frac{p_h^n - p_h^{n-1}}{\Delta t} - p_t^n, \vartheta^n \right) + (K(|\mathbf{s}_h^n|) \mathbf{s}_h^n - K(|\mathbf{s}^n|) \mathbf{s}^n, \eta^n) = 0. \quad (5.48)$$

We use (5.9), (5.48) to find that

$$\begin{aligned} \omega^n \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 &\leq (K(|\mathbf{s}_h^n|) \mathbf{s}_h^n - K(|\mathbf{s}^n|) \mathbf{s}^n, \mathbf{s}_h^n - \mathbf{s}^n) \\ &= (K(|\mathbf{s}_h^n|) \mathbf{s}_h^n - K(|\mathbf{s}^n|) \mathbf{s}^n, \eta^n) + (K(|\mathbf{s}_h^n|) \mathbf{s}_h^n - K(|\mathbf{s}^n|) \mathbf{s}^n, \zeta^n) \\ &= - \left(\frac{p_h^n - p_h^{n-1}}{\Delta t} - p_t^n, \vartheta^n \right) + (K(|\mathbf{s}_h^n|) \mathbf{s}_h^n - K(|\mathbf{s}^n|) \mathbf{s}^n, \zeta^n). \end{aligned} \quad (5.49)$$

Due to (5.12), Cauchy-Schwartz and triangle inequality, one has

$$\omega^n \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 \leq C ((\Delta t)^{-1} \|p_h^n - p_h^{n-1}\| + \|p_t^n\|) \|\vartheta^n\| + \mathcal{A} \|\zeta^n\|_{L^\beta(\Omega)}.$$

Using the fact that

$$\begin{aligned} (\Delta t)^{-1} \|p_h^n - p_h^{n-1}\| &= (\Delta t)^{-1} \left\| \int_{t_{n-1}}^{t_n} p_{h,t} dt \right\| \\ &\leq (\Delta t)^{-1} \int_{t_{n-1}}^{t_n} \|p_{h,t}\| dt \leq \sup_{[T/N, T]} \|p_{h,t}\| \leq \mathcal{B}, \end{aligned}$$

and

$$\|p_t^n\| \leq \sup_{[T/N, T]} \|p_t\| \leq \mathcal{B},$$

we obtain

$$\|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 \leq C\mathcal{B}(\omega^n)^{-1} \|\vartheta^n\| + \mathcal{A}(\omega^n)^{-1} \|\zeta^n\|_{L^\beta(\Omega)}.$$

It follows from (5.42) and (5.11) that

$$\begin{aligned} \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 &\leq C\mathcal{B}(\omega^n)^{-1} \left\{ \sum_{i=1}^n \Delta t \left(\mathcal{A} \|\zeta^i\|_{L^\beta(\Omega)} + \|\epsilon^i\|^2 \right) \right\}^{\frac{1}{2}} + \mathcal{A}(\omega^n)^{-1} \|\zeta^n\|_{L^\beta(\Omega)} \\ &\leq C\mathcal{A}^\lambda \mathcal{B} \left\{ \left(\mathcal{A} \sum_{i=1}^n \Delta t \|\zeta^i\|_{L^\beta(\Omega)} \right)^{\frac{1}{2}} + \Delta t \right\} + C\mathcal{A}^{\lambda+1} \|\zeta^n\|_{L^\beta(\Omega)}. \end{aligned} \quad (5.50)$$

Thus

$$\begin{aligned} \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)} &\leq C\mathcal{A}^{\frac{\lambda}{2} + \frac{1}{4}} \mathcal{B}^{\frac{1}{2}} h^{\frac{r+1}{4}} \left(\sum_{i=1}^n \Delta t \|\mathbf{s}^i\|_{r+1, \beta} d\tau \right)^{\frac{1}{4}} \\ &\quad + C\mathcal{A}^{\frac{\lambda}{2} + \frac{1}{2}} h^{\frac{r+1}{2}} \|\mathbf{s}^n\|_{r+1, \beta}^{\frac{1}{2}} + C\mathcal{A}^\lambda \mathcal{B} \sqrt{\Delta t}. \end{aligned} \quad (5.51)$$

The triangle inequality gives

$$\|\mathbf{u}_h^n - \mathbf{u}^n\|_{L^\beta(\Omega)} \leq C(\|\rho^n\|_{L^\beta(\Omega)} + \|\varrho^n\|_{L^\beta(\Omega)}).$$

Subtracting (3.11b) from (5.46b) and using $\mathbf{z}_h = (\rho^n)^{\beta-1}$ we have equation

$$(\rho^n, (\rho^n)^{\beta-1}) + (K(|\mathbf{s}_h^n|)\mathbf{s}_h^n - K(|\mathbf{s}^n|)\mathbf{s}^n, (\rho^n)^{\beta-1}) = 0.$$

Then according Cauchy-Schwartz inequality and Proposition 2.4,

$$\|\rho^n\|_{L^\beta(\Omega)} \leq C \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}.$$

Hence

$$\|\mathbf{u}_h^n - \mathbf{u}^n\|_{L^\beta(\Omega)} \leq C(\|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)} + \|\varrho^n\|_{L^\beta(\Omega)}). \quad (5.52)$$

Using (5.51) and (5.3) yield

$$\begin{aligned} \|\mathbf{u}_h^n - \mathbf{u}^n\|_{L^\beta(\Omega)} &\leq C\mathcal{A}^{\frac{\lambda}{2} + \frac{1}{4}} \mathcal{B}^{\frac{1}{2}} h^{\frac{r+1}{4}} \left(\sum_{i=1}^n \Delta t \|\mathbf{s}^i\|_{r+1, \beta} d\tau \right)^{\frac{1}{2}} \\ &\quad + C\mathcal{A}^{\frac{\lambda}{2} + \frac{1}{2}} h^{\frac{r+1}{2}} \|\mathbf{s}^n\|_{r+1, \beta}^{\frac{1}{2}} + Ch^{r+1} \|\mathbf{u}^n\|_{r+1, \beta} + C\mathcal{A}^\lambda \mathcal{B} \sqrt{\Delta t}. \end{aligned} \quad (5.53)$$

Therefore (5.44) follows from (5.51) and (5.53).

(ii) Thank to the regularity of solution we bound the right hand side of (5.49) using (5.16) instead of Cauchy-Schwartz inequality to obtain

$$\omega^n \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 \leq C\mathcal{B} \|\vartheta^n\| + \varepsilon \omega^n \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 + C(\varepsilon \omega^n)^{-1} \|\zeta^n\|_{L^\delta(\Omega)}^2.$$

or

$$\|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 \leq C\mathcal{B}(\omega^n)^{-1} \|\vartheta^n\| + \varepsilon \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 + C\varepsilon^{-1}(\omega^n)^{-2} \|\zeta^n\|_{L^\delta(\Omega)}^2.$$

Selecting $\varepsilon = \frac{1}{2}$, it follows from (5.42) and (5.11) that

$$\|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 \leq C\mathcal{A}^\lambda \mathcal{B} \left\{ \mathcal{A}^{\frac{\lambda}{2}} \left(\sum_{i=1}^n \Delta t \|\zeta^i\|_{L^\delta(\Omega)}^2 \right)^{\frac{1}{2}} + \Delta t \right\} + C\mathcal{A}^{2\lambda} \|\zeta^n\|_{L^\delta(\Omega)}^2.$$

Thus

$$\begin{aligned} \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)} &\leq C\mathcal{A}^{\frac{3\lambda}{4}} \mathcal{B}^{\frac{1}{2}} h^{\frac{r+1}{4}} \left(\sum_{i=1}^n \Delta t \|\mathbf{s}^i\|_{r+1,\delta}^2 d\tau \right)^{\frac{1}{4}} \\ &\quad + C\mathcal{A}^\lambda h^{\frac{r+1}{2}} \|\mathbf{s}^n\|_{r+1,\delta} + C\mathcal{A}^{\frac{\lambda}{2}} \mathcal{B} \sqrt{\Delta t}. \end{aligned}$$

This and (5.52) give us (5.37). We finish the proof. \square

6. Numerical results. In this section, we give a simple numerical result illustrating the convergence theory. We test the convergence of our method with the Forchheimer two term law. For simplicity, consider $g(s) = 1 + s$. Equation (2.4) $sg(s) = \xi$, $s \geq 0$ gives $s = \frac{-1+\sqrt{1+4\xi}}{2}$ and hence

$$K(\xi) = \frac{1}{g(s(\xi))} = \frac{2}{1 + \sqrt{1+4\xi}}.$$

Since we analyze a first order time discretization, we consider the lowest order mixed method. Here we use the lowest order Raviart-Thomas mixed finite element on the unit square in two dimensions. The chosen analytical solution is

$$\begin{aligned} p(x, t) &= e^{-5t} \left[\frac{1}{2}(x_1^2 + x_2^2) - \frac{1}{3}(x_1^3 + x_2^3) \right], \\ \mathbf{s}(x, t) &= \nabla p = e^{-5t}(x_1(1-x_1), x_2(1-x_2)), \\ \mathbf{u}(x, t) &= K(|\mathbf{s}|)\mathbf{s} = \frac{2\mathbf{s}(x, t)}{1 + \sqrt{1+4|\mathbf{s}(x, t)|}} \end{aligned}$$

for all $x \in \Omega, t \in [0, 1]$ where $x = (x_1, x_2), \Omega = [0, 1]^2$. The forcing term f is determined accordingly to the analytical solution by equation $p_t - \nabla \cdot \mathbf{u} = f$. Explicitly,

$$\begin{aligned} f(x, t) &= \frac{1}{2}(x_1^2 + x_2^2) - \frac{1}{3}(x_1^3 + x_2^3) - \frac{4e^{-5t}(1-x_1-x_2)}{1 + \sqrt{1+4|\mathbf{s}|}} \\ &\quad + \frac{4e^{-15t}}{|\mathbf{s}|(1 + \sqrt{1+4|\mathbf{s}|})^2 \sqrt{1+4|\mathbf{s}|}} \left[x_1^2(1-x_1)^2(1-2x_1) + x_2^2(1-x_2)^2(1-2x_2) \right]. \end{aligned}$$

We used FEniCS [23] to perform our numerical simulations. We divide the unit square into an $N \times N$ mesh of squares, each then subdivide into two right triangles using the UnitSquareMesh class in FEniCS. For each mesh, we solve the generalized Forchheimer equation numerically. The error control in each nonlinear solve is $\varepsilon = 10^{-6}$. Our problem is solved at each time level start at $t = 0$ until final time $T = 1$. At this time, we measured the L^2 -errors of pressure and L^β -errors of gradient of pressure and velocity. Here $\beta = 2 - a = 2 - \frac{\deg(g)}{\deg(g)+1} = \frac{3}{2}$. The numerical results are listed as the following table.

N	$\ p - p_h\ $	Rates	$\ s - s_h\ _{L^\beta(\Omega)}$	Rates	$\ \mathbf{u} - \mathbf{u}_h\ _{L^\beta(\Omega)}$	Rates
4	1.965e-01	-	2.505e-01	-	2.436e-01	-
8	1.011e-01	1.94	2.523e-01	0.99	2.504e-01	0.97
16	5.081e-02	1.98	2.525e-01	0.99	2.517e-01	0.99
32	2.542e-02	1.99	2.525e-01	1.00	2.519e-01	0.99
64	1.270e-02	2.00	2.524e-01	1.00	2.519e-01	1.00
128	6.351e-03	1.99	2.523e-01	1.00	2.519e-01	1.00
256	3.175e-03	2.00	2.521e-01	1.00	2.519e-01	1.00

Table 1. *Convergence study for generalized Forchheimer equation with zero flux on the boundary in 2D.*

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